

# A short return to simple AH-algebras with real rank zero

Huaxin Lin

## Abstract

Let  $A$  be a unital simple AH-algebra with stable rank one and real rank zero such that  $kx = 0$  for all  $x \in \ker \rho_A$ , the subgroup of infinitesimal elements in  $K_0(A)$ , and for the same integer  $k \geq 1$ . We show that  $A$  has tracial rank zero and is isomorphic to a unital simple AH-algebra with no dimension growth.

## 1 Introduction

One of the most successful aspects of operator algebras is the classification of simple separable amenable  $C^*$ -algebras, or otherwise known as the Elliott program. The program started with the classification of unital simple AT-algebras with real rank zero up to isomorphisms by their ordered  $K$ -theory (with the scale) by G. A. Elliott [4] which was preceded by Elliott's classification of AF-algebras ([3]) some fifteen years earlier. It was followed immediately by a number of earlier results. Then, Elliott and Gong ([5]) made a classification of AH-algebras with slow dimension growth and real rank zero by their scaled ordered  $K$ -theory. While the earlier results concentrated in the cases of real rank zero, attention later shifted to the case that  $C^*$ -algebras are not assumed to have real rank zero. One of the highlights of the program is the classification (up to isomorphisms) of unital simple AH-algebras with no dimension growth (see [6]). These are  $C^*$ -algebras whose real rank may not be zero (in fact these  $C^*$ -algebras have real rank one). However, the Elliott invariant this time involves not only the ordered  $K$ -groups but also tracial information. Just as one thought a complete classification for simple unital AH-algebras was possible, J. Villadsen provided examples of unital simple AH-algebras whose stable rank may be greater than one and examples of unital simple AH-algebras with stable rank one whose  $K_0$ -group may have perforation ([10] and [11]). It should be noted that unital simple AH-algebras with slow dimension growth have stable rank one and have weakly unperforated  $K_0$ -groups. On the other hand, it was proved in [7] that a unital simple AH-algebra with stable rank one, real rank zero and with weakly unperforated  $K_0$ -groups has tracial rank zero and therefore it is isomorphic to a unital simple AH-algebra with no dimension growth and with real rank zero. At that point, one might think the next goal would be the classification of unital simple AH-algebras of stable rank one and with weakly unperforated  $K_0$ -groups by their ordered  $K$ -groups and tracial information. However, A. Toms provided ([9]) an example of unital simple AH-algebra with stable rank one and with weakly unperforated  $K_0$ -group which is not isomorphic to a unital simple AH-algebra with slow dimension growth and with the same Elliott invariant.

The real rank of these AH-algebras are greater than zero. The purpose of this note is to show that, when a unital simple AH-algebra has real rank zero, then it is more likely classifiable by its Elliott invariant. Toms's first example is an inductive limit of homogeneous  $C^*$ -algebras whose spectra are contractive finite CW complexes. As we will show in this short note, this example cannot be made so that it has real rank zero. More precisely, we observe that a unital simple AH-algebra with real rank zero which is an inductive limit of homogeneous  $C^*$ -algebras whose spectra are contractive finite CW complexes is in fact a unital simple AH-algebra with

tracial rank zero. In particular, it is classifiable and isomorphic to a unital simple AH-algebra with no dimension growth. Using our earlier result in [7], we further show that if  $A$  is a unital simple AH-algebra with stable rank one and real rank zero so that the subgroup of infinitesimal elements of  $K_0(A)$  is a finite group (or zero), then  $A$  is isomorphic to a unital simple AH-algebra with no dimension growth and therefore it is classifiable by the Elliott invariant. We actually prove a slightly more general result.

**Acknowledgements:** This work was done in the summer 2011 when the author was in East China Normal University. It supported by East China Normal University and the Changjiang Lectureship there. This work was also supported by a grant from NSF.

## 2 The cases without torsion

Let  $A$  be a unital stably finite  $C^*$ -algebra. Denote by  $T(A)$  the tracial state space of  $A$ . If  $\tau \in T(A)$ , we will also use  $\tau$  for the trace  $\tau \otimes Tr$  on  $M_m(A)$ , where  $Tr$  is the standard trace on  $M_m$ , where  $m \geq 1$  is an integer. Let  $\rho_A : K_0(A) \rightarrow \text{Aff}(T(A))$  be the positive homomorphism defined by  $\rho_A([p])(\tau) = \tau(p)$  for all projections in  $M_m(A)$ ,  $m = 1, 2, \dots$

**Lemma 2.1.** *Let  $A = \lim_{n \rightarrow \infty} (A_n, \varphi_n)$  be a unital  $C^*$ -algebra, where  $A_n$  is a unital  $C^*$ -subalgebra and  $\varphi_n : A_n \rightarrow A_{n+1}$  is a unital homomorphism. Suppose that  $p, q \in A$  are two non-zero projections and there is an integer  $K \geq 1$  such that*

$$K\tau(q) < \tau(p) \text{ for all } \tau \in T(A).$$

*Then, there exist an integer  $n(1) \geq 1$ , two projections  $p', q' \in A_{n(1)}$  and an integer  $m_0 \geq n(1)$  such that  $\varphi_{n(1), \infty}(p')$  is unitarily equivalent to  $p$ ,  $\varphi_{n(1), \infty}(q')$  is unitarily equivalent to  $q$  and*

$$Kt(\varphi_{n(1), m}(q')) < t(\varphi_{n(1), m}(p'))$$

*for all tracial states  $t$  of  $A_m$  with  $m \geq m_0$ .*

*Proof.* Put  $B_n = \varphi_{n, \infty}(A_n)$ ,  $n = 1, 2, \dots$ . Without loss of generality, we may assume that there are  $p', q' \in A_{n(1)}$  for some  $n(1)$  such that

$$\varphi_{n(1), \infty}(p') = p \text{ and } \varphi_{n(1), \infty}(q') = q.$$

Suppose that, for some increasing subsequence  $\{n(k)\}$ ,

$$Kt_k(\varphi_{n(1), n(k)}(q')) \geq t_k(\varphi_{n(1), n(k)}(p')) \tag{e.2.1}$$

for some tracial states  $t_k$  of  $A_{n(k)}$ . By a result of Choi and Effros ([2]), there exists a contractive completely positive linear map  $L_k : B_k \rightarrow A_k$  such that  $\varphi_{k, \infty} \circ L_k = \text{id}_{B_k}$ ,  $k = 1, 2, \dots$ . Note that, if  $m \geq n(1)$ ,

$$\lim_{k \rightarrow \infty} \|\varphi_{m, n(k)}[L_m(\varphi_{n(1), \infty}(b)) - \varphi_{n(1), m}(b)]\| = 0 \tag{e.2.2}$$

for all  $b \in A_{n(1)}$ . In particular, it holds for  $m = n(2)$ .

Define  $s'_k : B_{n(k)} \rightarrow \mathbb{C}$  by  $s'_k(b) = t_k \circ L_{n(k)}(b)$  for all  $b \in B_{n(k)}$ ,  $k = 1, 2, \dots$ . Then  $s'_k$  is a state on  $B_{n(k)}$ . Let  $s_k$  be a state of  $A$  which extends  $s'_k$ . Let  $\tau$  be a weak limit of  $\{t_k\}$ . It follows from (e.2.2) that  $\tau$  is a tracial state on  $B_{n(1)}$ . One would have, by (e.2.1),

$$K\tau(q) \geq \tau(p).$$

A contradiction. □

**Definition 2.2.** Recall that a unital  $C^*$ -algebra  $A$  is said to be AH-algebra if  $A = \lim_{n \rightarrow \infty} (A_n, \varphi_n)$ , where  $A_n = P_n M_{r(n)}(C(X_n)) P_n$ ,  $X_n$  is a finite CW complex,  $r(n) \geq 1$  is an integer and  $P_n \in M_{r(n)}(C(X_n))$  is a projection. We also assume that  $\varphi_n : A_n \rightarrow A_{n+1}$  is unital. In what follows, we call  $X_n$  the spectrum of  $A_n$ .

**Theorem 2.3.** *Let  $A = \lim_{n \rightarrow \infty} (A_n, \varphi_n)$  be a unital simple AH-algebra such that the spectrum of  $A_n$  is a finite disjoint union of contractive spaces and  $\varphi_n : A_n \rightarrow A_{n+1}$  is a unital homomorphism. Suppose also that  $A$  has the real rank zero. Then  $A$  has tracial rank zero.*

*Proof.* Let  $p, q \in A$  be two projections such that

$$\tau(q) < \tau(p)$$

for all  $\tau \in T(A)$ . We will show that  $q \lesssim p$ .

We assume that  $A_n = M_{r(n)}(C(X_n))$ , where  $X_n$  is a disjoint union of contractive spaces. We may also assume that there are two projections  $p', q' \in A_n$  for some  $n \geq 1$  such that  $\varphi_{n,\infty}(p') = p$  and  $\varphi_{n,\infty}(q') = q$ . Moreover, by 2.1, we may assume that

$$t(q') < t(p')$$

for all tracial states  $t$  of  $A_n$ . Since  $X_n$  is a disjoint union of contractive spaces, all projections are trivial. Therefore in  $A_n$ ,  $q' \lesssim p'$ . It follows that  $q \lesssim p$ . This implies that  $K_0(A)$  is weakly unperforated.

We will show that projections in  $A$  have cancellation property. Suppose that  $p, q \in A$  and  $p$  and  $q$  are equivalent. As above, there are  $p', q' \in A_n$  such that  $\varphi_{n,\infty}(p') = p$  and  $\varphi_{n,\infty}(q') = q$ . Moreover, we may assume that  $p'$  and  $q'$  are equivalent. So in  $A_n$ ,  $1 - p'$  and  $1 - q'$  have the same rank at each point. Note that, since  $X_n$  is a disjoint union of contractive spaces,  $1 - p'$  and  $1 - q'$  are trivial projections. It follows that  $1 - p'$  and  $1 - q'$  are equivalent. It follows that  $1 - p$  and  $1 - q$  are equivalent in  $A$ .

Since  $A$  is also assumed to have real rank zero, by (the proof of) part (3) of III.2.4 of [1],  $A$  has stable rank one. It follows from [7] that  $A$  has tracial rank zero. □

**Corollary 2.4.** *Let  $A = \lim_{n \rightarrow \infty} (A_n, \varphi_n)$  be a unital simple AH-algebra such that  $\ker \rho_{A_n} = \{0\}$ . Suppose that  $A$  has real rank zero and stable rank one. Then  $A$  has tracial rank zero and is isomorphic to a unital simple AH-algebra with no dimension growth.*

*Proof.* The proof is the same as that of 2.3. Note that since  $\ker \rho_{A_n} = \{0\}$ , two projections  $p, q$  in  $M_m(A_n)$  with the same rank must give the same element in  $K_0(A_n)$ . □

**Lemma 2.5.** *Let  $A$  be a unital AH-algebra and  $x \in \rho_A(K_0(A))$  such that  $x > 0$ . Then, there is a projection  $e \in M_m(A)$  for some integer  $m \geq 1$  such that  $\rho_A([e]) = x$ .*

*Proof.* Note that  $M_m(A)$  is also a unital AH-algebra for any integer  $m \geq 1$ . We may assume that there are two projections  $p, q \in A$  such that

$$\rho([p] - [q]) = x.$$

We write  $A = \lim_{n \rightarrow \infty} (A_n, \varphi_n)$ , where  $A_n = P_n M_{r(n)}(C(X_n)) P_n$ ,  $X_n$  is a finite CW complex, and  $r(n) \geq 1$  is an integer and  $P_n \in M_{r(n)}(C(X_n))$  is a projection. We may assume that there is an integer  $n \geq 1$  and projections  $p', q' \in A_n$  such that  $\varphi_{n,\infty}(p') = p$  and  $\varphi_{n,\infty}(q') = q$ . By 2.1, we may also assume that

$$t(q') < t(p') \tag{e 2.3}$$

for all tracial states  $t$  of  $A_n$ . There is an integer  $K(n) \geq 1$  and there are trivial projections  $p_0, q_0 \in M_{K(n)}(A_n)$  such that  $p_0$  has the same rank at each point of  $X$  as that of  $p'$ , and  $q_0$  has the same rank at each point of  $X$  as that of  $q'$ . Therefore

$$t(p_0) = t(p') \text{ and } t(q_0) = t(q') \quad (\text{e 2.4})$$

for all tracial states  $t$  of  $A_n$ . It follows that

$$\tau(\varphi_{n,\infty}(p_0)) = \tau(p) \text{ and } \tau(\varphi_{n,\infty}(q_0)) = \tau(q) \quad (\text{e 2.5})$$

for all  $\tau \in T(A)$ . On the other hand, by (e 2.3) and (e 2.4), since both  $p_0$  and  $q_0$  are trivial, there is a partial isometry  $v_0 \in A_n$  such that

$$v_0^*v_0 = q_0 \text{ and } v_0v_0^* \leq p_0.$$

Let  $v = \varphi_{n,\infty}(v_0)$  and let  $e = p_0 - vv^*$ . Then  $e \in M_m(A)$  is a non-zero projection and, by (e 2.5),

$$\rho_A([e]) = x.$$

□

From the above and combining the result in [7], one has the following corollary:

**Corollary 2.6.** *Let  $A$  be a unital simple AH-algebra with real rank zero and stable rank one. Suppose that  $\ker \rho_A = \{0\}$ . Then  $A$  has tracial rank zero.*

Note that, 2.6 is not a generalization of 2.4. In fact that a simple AF-algebra may have non-zero infinitesimal elements in its  $K_0$ -group.

We will prove a much more general result that will allow non-zero  $\ker \rho_A$ .

### 3 The case that $\rho_A(K_0(A))$ is torsion

**Lemma 3.1.** *Let  $A$  be a unital simple AH-algebra with stable rank one and with real rank zero. Let  $K \geq 1$  be an integer. Suppose that for any  $x \in \ker \rho_A$ ,  $Kx = 0$ . Suppose that  $p, q \in M_m(A)$  are two projections such that  $(K+2)\tau(q) < \tau(p)$  for all  $\tau \in T(A)$ . Then*

$$[q] \leq [p].$$

*Proof.* Since  $A$  has real rank zero, by a result of S. Zhang ([12], see also 9.4 of [8]), there exist mutually orthogonal projections  $e_1, e_2, \dots, e_{K+1} \in pAp$  such that  $e_i$  is equivalent to  $e_1$  for  $i = 1, 2, \dots, K$ ,  $e_{K+1} \lesssim e_1$  and  $p = \sum_{i=1}^{K+1} e_i$ . Write  $A = \lim_{n \rightarrow \infty} (A_n, \varphi_n)$ , where  $A_n = P_n M_{r(n)}(C(X)) P_n$ ,  $X$  is a finite CW complex,  $r(n) \geq 1$  is an integer and  $P_n \in M_{r(n)}(C(X))$  is a projection. Without loss of generality, we may assume that there are projections

$$p', q', e'_1, e'_2, \dots, e'_{K+1} \in M_m(A_n)$$

such that  $\varphi_{n,\infty}(p') = p$ ,  $\varphi_{n,\infty}(q') = q$ ,  $\varphi_{n,\infty}(e'_i) = e_i$ ,  $i = 1, 2, \dots, K+1$ . We may also assume that  $e'_i \leq p'$ ,  $i = 1, 2, \dots, K+1$ ,  $e'_1, e'_2, \dots, e'_{K+1}$  are mutually orthogonal,  $e'_i$  is equivalent to  $e'_1$  in  $M_m(A_n)$ ,  $i = 1, 2, \dots, K$ ,  $e'_{K+1} \lesssim e'_1$  in  $M_m(A_n)$  and  $\sum_{i=1}^{K+1} e'_i = p'$ . Moreover, by 2.1, we may also assume that

$$(K+2)t(q') < t(p') \leq (K+1)t(e'_1)$$

for all tracial states  $t$  of  $A_n$ . Note also, we have

$$(K+1)t(q') < Kt(e'_1) \quad (\text{e 3.6})$$

for all tracial states  $t$  of  $A_n$ .

There are trivial projections  $q'', e_1'' \in M_R(A_n)$  for some  $R \geq 1$  such that

$$t(q'') = t(q') \quad \text{and} \quad t(e_1'') = t(e_1')$$

for all tracial states  $t \in T(A_n)$ . Let  $\bar{q}, \bar{e} \in M_R(A)$  such that  $\varphi_{n,\infty}(q'') = \bar{q}$  and  $\varphi_{n,\infty}(e_1'') = \bar{e}$ . It follows that

$$\tau(\bar{q}) = \tau(q) \quad \text{and} \quad \tau(\bar{e}) = \tau(e_1)$$

for all  $\tau \in T(A)$ . Therefore

$$[\bar{q}] - [q] \quad \text{and} \quad [\bar{e}] - [e_1] \quad \text{are in} \quad \ker \rho_A.$$

It follows from the assumption that

$$K[q] = K[\bar{q}] \quad \text{and} \quad K[\bar{e}] = K[e_1]. \quad (\text{e 3.7})$$

On the other hand, by (e 3.6), since both  $q''$  and  $e_1''$  are trivial,

$$q'' \lesssim e_1''. \quad (\text{e 3.8})$$

By (e 3.7) and (e 3.8),

$$K[q] = K[\bar{q}] \leq K[e_1] \leq p. \quad (\text{e 3.9})$$

Since  $A$  also has stable rank one,

$$q \lesssim e_1 + e_2 + \cdots + e_K \leq p. \quad (\text{e 3.10})$$

□

The following theorem was proved in [7].

**Theorem 3.2.** *Let  $A$  be a unital simple AH-algebra with stable rank one and real rank zero. Then, for any  $\epsilon > 0$ ,  $\sigma > 0$  and any finite subset  $\mathcal{F} \subset A$ , there exists a non-zero projection  $p \in A$  and a finite dimensional  $C^*$ -subalgebra  $B \subset A$  with  $1_B = p$  such that*

$$\|px - xp\| < \epsilon \quad \text{for all } x \in \mathcal{F}, \quad (\text{e 3.11})$$

$$\text{dist}(pxp, B) < \epsilon \quad \text{for all } x \in \mathcal{F} \quad \text{and} \quad (\text{e 3.12})$$

$$\tau(1 - p) < \sigma \quad \text{for all } \tau \in T(A). \quad (\text{e 3.13})$$

**Corollary 3.3.** *Let  $A$  be a unital simple AH-algebra with stable rank one and real rank zero. Let  $e_1, e_2 \in A$  be two projections such that*

$$\tau(e_1) > \tau(e_2) \quad \text{for all } \tau \in T(A). \quad (\text{e 3.14})$$

*Then, for any  $\epsilon > 0$ ,  $\sigma > 0$  and any finite subset  $\mathcal{F} \subset A$ , there exists a non-zero projection  $p \in A$  and a finite dimensional  $C^*$ -subalgebra  $B \subset A$  with  $1_B = p$  such that*

$$\|px - xp\| < \epsilon \quad \text{for all } x \in \mathcal{F}, \quad (\text{e 3.15})$$

$$\text{dist}(pxp, B) < \epsilon \quad \text{for all } x \in \mathcal{F} \quad \text{and} \quad (\text{e 3.16})$$

$$\tau(1 - p) < \sigma \quad \text{for all } \tau \in T(A). \quad (\text{e 3.17})$$

*Moreover, there are projections  $e_{j,0} \in (1 - p)A(1 - p)$  and  $e_{j,1} \in B$ ,  $j = 1, 2$ , such that*

$$\|e_{j,0} + e_{j,1} - e_j\| < \min\{1/2, \epsilon\}, \quad j = 1, 2, \quad (\text{e 3.18})$$

$$t(e_{1,1}) > t(e_{2,1}) \quad \text{for all tracial states of } B \quad \text{and} \quad (\text{e 3.19})$$

$$\tau(e_{1,0}) > \tau(e_{2,0}) \quad \text{for all } \tau \in T(A). \quad (\text{e 3.20})$$

*Proof.* Note that  $A$  is separable. Let  $\epsilon > 0$ ,  $\sigma > 0$  and let  $\mathcal{F} \subset A$  be a finite subset. Let  $\{x_1, x_2, \dots\}$  be a dense subset of  $A$ . Let  $\mathcal{F}_n = \mathcal{F} \cup \{e_1, e_2\} \cup \{x_1, x_2, \dots, x_n\}$  and let  $\epsilon_n = \epsilon/2^n$  and  $\sigma_n = \sigma/2^n$ ,  $n = 1, 2, \dots$ . By 3.2, there exists a sequence of projections  $p_n \in A$  and finite dimensional  $C^*$ -subalgebras  $B_n \subset A$  with  $1_{B_n} = p_n$  such that

$$\|p_n x - x p_n\| < \epsilon/2^n \text{ for all } x \in \mathcal{F}_n, \quad (\text{e 3.21})$$

$$\text{dist}(p_n x p_n, B_n) < \epsilon/2^n \text{ for all } x \in \mathcal{F}_n \text{ and } \tau(1 - p_n) < \sigma/2^n \quad (\text{e 3.22})$$

$n = 1, 2, \dots$ . For all sufficiently large  $n$ , there are projections  $e^{(j,1,n)} \in B_n$  and  $e^{(j,0,n)} \in (1 - p_n)A(1 - p_n)$  such that

$$\|e^{(j,1,n)} + e^{(j,0,n)} - e_j\| < \epsilon/2^{n-2}, \quad (\text{e 3.23})$$

$j = 1, 2$  and  $n = 1, 2, \dots$ . We also have

$$\lim_{n \rightarrow \infty} \|p_n e_j p_n - e^{(j,1,n)}\| = 0, \quad j = 1, 2. \quad (\text{e 3.24})$$

Suppose that, for a subsequence  $\{n_k\}$ ,

$$t_k(e^{(2,1,n_k)}) \leq t_k(e^{(1,1,n_k)}) \quad (\text{e 3.25})$$

for some tracial state  $t_k$  of  $B_{n_k}$ . Let  $s'_k : p A p \rightarrow \mathbb{C}$  be a state which extends  $t_k$ . Define  $s_k : A \rightarrow \mathbb{C}$  by  $s_k(a) = s'_k(p a p)$  for all  $a \in A$ . Then  $s_k$  is a state on  $A$ . Let  $t$  be a weak limit of  $\{s_k\}$ . One checks that  $t$  is a tracial state on  $A$ . Then (e 3.25) and (e 3.24) imply that

$$t(e_1) \leq t(e_2)$$

which contradicts with (e 3.14).

It follows that, for all sufficiently large  $n$ ,

$$t(e^{(2,1,n)}) > t(e^{(1,1,n)}) \text{ for all } t \in T(B_n). \quad (\text{e 3.26})$$

The lemma follows. □

**Theorem 3.4.** *Let  $A$  be a unital simple AH-algebra with stable rank one and real rank zero. Suppose that there is an integer  $K \geq 1$  such that, for any  $x \in \ker \rho_A$ ,  $Kx = 0$ . Then  $A$  has tracial rank zero. Moreover,  $A$  is isomorphic to a unital simple AH-algebra with slow dimension growth.*

*Proof.* We may assume that  $A$  is infinite dimensional. We will show that  $K_0(A)$  is weakly unperforated. It then follows from [7] that  $A$  has tracial rank zero.

It suffices to show the following: If  $p, q \in M_m(A)$  are two non-zero projections for some integer  $m \geq 1$  and

$$\tau(p) > \tau(q) \text{ for all } \tau \in T(A),$$

then  $q \lesssim p$ .

To prove this, we note that  $M_m(A)$  is also a unital simple AH-algebra with stable rank one, real rank zero and  $K_0(M_m(A)) = K_0(A)$ , so, to simplify the notation, without loss of generality, we may assume that  $p, q \in A$ .

Let

$$d_1 = \inf\{\tau(p) - \tau(q) : \tau \in T(A)\}.$$

Since  $A$  is simple,  $d_1 > 0$ . Since  $A$  is an infinite dimensional simple  $C^*$ -algebra with real rank zero,  $pAp$  is also an infinite dimensional simple  $C^*$ -algebra with real rank zero. It follows that there is a non-zero projection  $e \in pAp$  such that

$$\tau(e) < d_1/2 \text{ for all } \tau \in T(A). \quad (\text{e 3.27})$$

Put

$$d_2 = \inf\{\tau(e) : \tau \in T(A)\}. \quad (\text{e 3.28})$$

Note that  $d_2 > 0$ . Put  $p_0 = p - e$ . Then

$$\tau(p_0) > \tau(q) \text{ for all } \tau \in T(A). \quad (\text{e 3.29})$$

Let

$$\mathcal{F} = \{p, q, e, p_0\}.$$

It follows from 3.2 and 3.3 that, there exists a projection  $E \in A$  and a finite dimensional  $C^*$ -subalgebra  $B$  with  $1_B = E$  such that

$$\|Ex - xE\| < \frac{d_2}{64K} \text{ for all } x \in \mathcal{F}; \quad (\text{e 3.30})$$

$$\text{dist}(ExE, B) < \frac{\min\{d_2, 1\}}{64K} \text{ for all } x \in \mathcal{F} \text{ and} \quad (\text{e 3.31})$$

$$\tau(1 - E) < \frac{d_2}{64K} \text{ for all } \tau \in T(A). \quad (\text{e 3.32})$$

Moreover, there are projections  $p_{0,1}, q_1 \in B$  and  $p_{0,0}, q_0 \in (1 - E)A(1 - E)$  such that

$$\|p_{0,1} + p_{0,0} - p_0\| < \frac{1}{16K}, \quad (\text{e 3.33})$$

$$\|q_1 + q_0 - q\| < \frac{1}{16K} \text{ and} \quad (\text{e 3.34})$$

$$t(q_1) < t(p_{0,1}) \quad (\text{e 3.35})$$

for all tracial state  $t$  of  $B$ . It follows that, in  $B$ ,

$$q_1 \lesssim p_{0,1}. \quad (\text{e 3.36})$$

We compute, by (e 3.28) and (e 3.32) that

$$(K + 2)\tau(q_0) < \tau(e) \text{ for all } \tau \in T(A). \quad (\text{e 3.37})$$

It follows from 3.1 that

$$q_0 \lesssim e. \quad (\text{e 3.38})$$

Combining (e 3.36) and (e 3.38), we obtain that

$$q_1 + q_0 \lesssim p_{0,1} + e.$$

But, by (e 3.33) and (e 3.34),

$$[q_1 + q_0] = [q] \text{ and } p_{0,1} \leq p_0.$$

Therefore

$$q \lesssim p.$$

□

**Corollary 3.5.** *Let  $A$  be a unital simple AH-algebra with stable rank one and real rank zero. Suppose that  $\ker \rho_A$  is finite. Then  $A$  has tracial rank zero and  $K_0(A)$  is weakly unperforated.*

## 4 Concluding remarks

**Remark 4.1.** Theorem 2.3 shows that that Toms' example (as in [9]) could not occur under the assumption that  $A$  has real rank zero.

**Remark 4.2.** More can be said in Theorem 2.3. It is clear that that, in Theorem 2.3, it suffices to assume that every  $A_n$  has the property that all projections in the matrix algebras of  $A_n$  are unitarily equivalent to those constant projections. So it allows  $A_n$  to have non-zero  $K_1$ -groups.

Let  $A_n = P_n M_{r(n)}(C(X_n)) P_n$ , where  $X_n$  is a finite CW complex,  $r(n) \geq 1$  is an integer and  $P_n \in M_{r(n)}(C(X_n))$ . Suppose that  $Y_n$  is another finite CW complex with covering dimension  $hd(X_n)$  which has the same homotopy type of that of  $X_n$ . Suppose that

$$\liminf_{n \rightarrow \infty} \left( \sup_{x \in X_n} \frac{hd(X_n)}{\text{Rank} P_n(x)} \right) = 0.$$

Suppose also that  $A = \lim_{n \rightarrow \infty} A_n$  is a unital simple  $C^*$ -algebra with real rank zero. Then, from the proof of 2.3, one can show that  $A$  has tracial rank zero and stable rank one. In other words, if  $A$  is homotopically slow dimension growth and is of real rank zero and stable rank one, then  $A$  is isomorphic to a unital simple AH-algebra with no dimension growth. One should note that, as in [9], without the assumption of real rank zero, the Cuntz semigroup of  $A$  could be very different from those of unital simple AH-algebras with slow dimension growth.

**Remark 4.3.** From the proof of 3.4, one sees that Theorem 3.4 holds if the assumption on  $\ker \rho_A$  is replaced by the conclusion of 3.1, i.e., there is an integer  $K \geq 1$  such that for any pair of projections  $p, q \in M_m(A)$  (for any integer  $m \geq 1$ ),  $K\tau(p) \leq \tau(q)$  for all  $\tau \in T(A)$  implies  $p \lesssim q$ .

## References

- [1] B. Blackadar and D. Handelman, *Dimension functions and traces on  $C^*$ -algebras*, J. Funct. Anal. **45** (1982), 297–340.
- [2] M-D. Choi and E. G. Effros, *The completely positive lifting problem for  $C^*$ -algebras*, Ann. of Math. **104** (1976), 585 – 609
- [3] G. A. Elliott, *On the classification of inductive limits of sequences of semisimple finite-dimensional algebras*, J. Algebra **38** (1976), 29–44.
- [4] G. A. Elliott, *On the classification of  $C^*$ -algebras of real rank zero*, J. Reine Angew. Math. **443** (1993), 179–219.
- [5] G. A. Elliott and G. Gong, *On the classification of  $C^*$ -algebras of real rank zero. II*, Ann. of Math. **144** (1996), 497–610.
- [6] G. A. Elliott, G. Gong and L. Li, *On the classification of simple inductive limit  $C^*$ -algebras. II, The isomorphism theorem*, Invent. Math. **168** (2007), 249–320.
- [7] H. Lin, *Simple AH-algebras of real rank zero*, Proc. Amer. Math. Soc. **131** (2003), 3813–3819
- [8] H. Lin, *Approximate homotopy of homomorphisms from  $C(X)$  into a simple  $C^*$ -algebra*, Mem. Amer. Math. Soc. **205** (2010), no. 963, vi+131 pp.



- [9] A. S. Toms, Andrew *On the classification problem for nuclear  $C^*$ -algebras*, Ann. of Math. **167** (2008), 1029–1044.
- [10] J. Villadsen, *Simple  $C^*$ -algebras with perforation*, J. Funct. Anal. **154** (1998), 110–116.
- [11] J. Villadsen, *On the stable rank of simple  $C^*$ -algebras*, J. Amer. Math. Soc. **12** (1999), 1091C1102.
- [12] S. Zhang, *Matricial structure and homotopy type of simple  $C^*$ -algebras with real rank zero*, J. Operator Theory **26** (1991), 283–312.

hlin@uoregon.edu